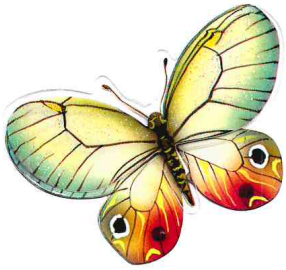
Der anisotrope Homöomorphismus des Allgemeinen Gehirnfurchen-Krümmungstheorie und ihre Implikationen für die Appersezeption des Quantenuniversums.

oder...

Die traumsiebenden SCHWARZSCHILD-Lösungen der Einstein-Cajal'schen Zentralfurchen-gleichungen in der Topologie der convex-fraktalen Mannigfaltigkeit des hirnspezifischen Sulci.



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Article

On Exact Solutions to the Einstein Field Equations

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Being in relation to time is like the macrocosm to the microcosm, connected through a Quantum world-bridge-lace
Dante'sche Teufelsdreieckigkeit
Fractal in a quantum phase space
Albert Einstein
$\Phi(r,t) = \iiint_{Brain} G(r,t) \cdot P(r',t)\, dV(r')$
IMAGINATION IS MORE IMPORTANT THAN KNOWLEDGE

Contents

List of Symbols 6

I. Space-Time Singularity – Mass Point 10

1. On the gravitational field of a mass point according to Einstein's theory 11

II. Extended Body in Space-Time – Sphere 22

2. On the gravitational field of a sphere of incompressible fluid according to Einstein's theory 23

III. Literature 38

References 39

List of Symbols

f_1, f_2, f_3, f_4	real-valued functions of only the first component, $x_1 (= x)$, of the four-dimensional position vector
$g_{\mu\nu}$	Mathematically not exactly called a metric tensor or metric for short; A (symmetric) matrix, consisting of a set of 16 (in the case of symmetry 10 independent) real-valued functions
$G_{\mu\nu}$	EINSTEIN tensor, see `https://en.wikipedia.org/wiki/Einstein_tensor`
k^2	GAUSSian gravitational constant, see `https://en.wikipedia.org/wiki/Gaussian_gravitational_constant`
K	defined according to Equation 2.28
M	mass of sphere
P_a	denotes the "externally measured" radius
r	radius of sphere
R	auxiliary quantity with the unit meter
s	a scalar parameter of motion (e.g. the proper time)
$\mathrm{d}s^2$	line element, see `https://en.wikipedia.org/wiki/Line_element`
T	RICCI scalar or scalar curvature, see `https://en.wikipedia.org/wiki/Scalar_curvature`
$T_{\mu\nu}$	RICCI curvature tensor, see `https://en.wikipedia.org/wiki/Ricci_curvature`
v	velocity or falling speed, respectively
V	Volume of sphere
x_1, x_2, x_3, x_4	Components of a four-dimensional position vector (or a 4-position vector) in four-dimensional space (a four-dimensional space-time)

α	constant of integration; depending on the size of the mass at the origin of the sphere, see Equation 2.35. In modern notation this quantity is called the SCHWARZSCHILD radius, see `https://en.wikipedia.org/wiki/Schwarzschild_radius`
γ	defined according to Equation 2.11
Γ	CHRISTOFFEL symbol, see `https://en.wikipedia.org/wiki/Christoffel_symbols`
ζ	constant of integration, see Equation 2.34
$\varkappa$	$= 8\pi \cdot k^2$
λ	constant of integration; it turns out that $\lambda = 0$, otherwise physically meaningless solutions result
ρ_0	density of fluid (within the sphere under consideration)
χ	variable, introduced with Equation 2.29
ω	angular velocity
$_a$	outside, on the surface of the sphere (from German *a*, abbreviation for "außen"); may be understood in the sense of a surrounding (outer) spherical boundary, i.e. the event horizon

Abstract

This fundamental memoir contains the translated and original form of the solution of SCHWARZSCHILD's problem. It is regular in the whole space-time, with the only exception of the origin of the spatial co-ordinates; consequently, it leaves no room for the science fiction of the black holes. (In the centuries of the decline of the Roman Empire people said: "Graecum est, non legitur"...).

For why you do exist, when nobody can apprehend you?

Part I.

Space-Time Singularity – Mass Point

1. On the gravitational field of a mass point according to Einstein's theory

§1. In his work on the motion of the perihelion of Mercury (see [3]) Mr. Einstein has posed the following problem:

Let a point move according to the requirement:

$$\left.\begin{aligned} &\delta \cdot \int \mathrm{d}s = 0\,, \\ &\text{where} \\ &\qquad \mathrm{d}s = \sqrt{\sum \mathrm{g}_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu} \qquad \mu,\nu = 1,2,3,4\,, \end{aligned}\right\} \tag{1.1}$$

$\mathrm{g}_{\mu\nu}$ are functions of the variables x, and, in the framework of this variation, these variables are to be fixed in the start and the end of the path of integration. Hence, in short, this point moves along a geodesic line in a manifold, which is characterized by the line element $\mathrm{d}s$.

Performing the variation, the equations of motion of this point are obtained, which read as

$$\frac{\mathrm{d}^2 x_\alpha}{\mathrm{d}s^2} = \sum_{\mu,\nu} \Gamma^\alpha_{\mu\nu} \cdot \frac{\mathrm{d}x_\mu}{\mathrm{d}s}\frac{\mathrm{d}x_\nu}{\mathrm{d}s}\,, \quad \alpha,\beta = 1,2,3,4 \tag{1.2}$$

where

$$\Gamma^\alpha_{\mu\nu} = -\frac{1}{2} \cdot \sum_\beta g^{\alpha\beta} \cdot \left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right)\,, \tag{1.3}$$

and where $\mathrm{g}^{\alpha\beta}$ means the contravariant and normalized sub-determinant to $\mathrm{g}_{\alpha\beta}$ in the determinant $|\,\mathrm{g}_{\mu\nu}\,|$ (i.e. the matrix inverse or invertible matrix).

According to Einstein's theory, this is exactly the case when the movement is that of a massless point in the gravitational field of a mass located at the point $x_1 = x_2 = x_3 = 0$, i.e. when the "components of the gravitational field" Γ satisfy

everywhere, with exception of the point $x_1 = x_2 = x_3 = 0$, the "field equations"

$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} = 0 \tag{1.4}$$

and if at the same time the "determinant equation"

$$| \, \mathrm{g}_{\mu\nu} \, | = -1 \tag{1.5}$$

is satisfied, i.e. is symmetrical with respect to the indices μ and ν.

These field equations in connection with the determinant equation exhibit the fundamental property that their form remains unchanged in the framework of substitution of any other variables instead of x_1, x_2, x_3, x_4, if only the substitution-determinant still equals 1.

If x_1, x_2, x_3 be curvilinear coordinates and if x_4 be the time, further if the mass located at the origin of the coordinate system be unchanged with time, and if also the motion at infinity be uniform and linear, then according to Mr. Einstein, [4], the following requirements are also to be satisfied:

1. All the components are independent of the time coordinate x_4.

2. The equations $g_{\rho 4} = g_{4\rho} = 0$ apply exactly for $\rho = 1, 2, 3$.

3. The solution is spatially symmetric, i.e. in some way scale invariant around the origin of the coordinate system in that sense that the same solution results after the orthogonal transformation (rotation) of x_1, x_2, x_3 is performed.

4. The $\mathrm{g}_{\mu\nu}$ vanish at infinity, with the exception of the following four non-zero limit values:
$$g_{44} = 1 \, , \quad g_{11} = g_{22} = g_{33} = -1 \, .$$

The challenge is to find a dedicated line element with such coefficients that the field equations, the determinant equation, and the four requirements item 1 to item 4 are fulfilled simultaneously.

§2. Mr. Einstein showed that this problem as a first approximation leads to Newton's law and that the second approximation reproduces the well-known anomaly in the movement of Mercury's perihelion correctly. The following calculation provides the rigorously solution of the problem. It's always a pleasure to have mathematically sound solutions of a simple form. What is more important is that the calculation results in a uniquely determined solution, of which Mr.

EINSTEIN's treatment still left some doubts and which, as it presents itself below, may hardly be proven with such an approximation method. So the following lines lead to the fact that Mr. EINSTEIN's result will shine forth in more intensified purity.

§3. If we call the time t and the rectangular coordinates (i.e. the Cartesian coordinates) x, y, z, the most general form of the line element $\mathrm{d}s$, which meets the requirements according to item 1 till item 3 is obviously the following one:

$$\mathrm{d}s^2 = F \cdot \mathrm{d}t^2 - G \cdot \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) - H \cdot (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)^2$$

where F, G, H are functions of $r = \sqrt{x^2 + y^2 + z^2}$.

The condition item 4 requires that for $r = \infty$: $F = G = 1$ and $H = 0$.
Moving to spherical polar coordinates (radius r, inclination ϑ, azimuth φ) according to

$$\begin{aligned} x &= r \cdot \sin(\vartheta) \cdot \cos(\varphi) \\ y &= r \cdot \sin(\vartheta) \cdot \sin(\varphi) \\ z &= r \cdot \cos(\vartheta)\,, \end{aligned}$$

the same line element takes the form

$$\begin{aligned} \mathrm{d}s^2 &= F \cdot \mathrm{d}t^2 - G \cdot \left(\mathrm{d}r^2 + r^2\,\mathrm{d}\vartheta^2 + r^2 \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right) - H \cdot r^2\,\mathrm{d}r^2 \\ &= F \cdot \mathrm{d}t^2 - \left(G + H \cdot r^2\right)\mathrm{d}r^2 - Gr^2 \cdot \left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right)\,. \end{aligned} \tag{1.6}$$

Meanwhile, the volume element in spherical polar coordinates is equal to $r^2\,\mathrm{d}r\sin(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$, the determinant of transformation of the old as well as the new coordinates $r^2 \cdot \sin(\vartheta)$ is different from 1. So the field equations would not exist in unchanged form if one would calculate with these spherical polar coordinates and one would have to carry out a complicated transformation. However, a simple trick allows this difficulty to circumnavigate elegantly. We set

$$\left.\begin{aligned} x_1 &= \frac{r^3}{3} \\ x_2 &= -\cos(\vartheta) \\ x_3 &= \varphi\,. \end{aligned}\right\} \tag{1.7}$$

Then the following holds for the volume element: $r^2\,\mathrm{d}r\sin(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \mathrm{d}x_1 \cdot \mathrm{d}x_2 \cdot \mathrm{d}x_3$. These new variables *therefore are spherical polar coordinates of the determinant 1.* They have the advantageous of the classical spherical coordinates to solve the problem and furthermore, in using them, the field equations remain unchanged if one adds in addition $t = x_4$.

With these new spherical polar coordinates the line element reads as:

$$\begin{aligned} \mathrm{d}s = F \cdot \mathrm{d}x_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right) \mathrm{d}x_1^2 \\ - Gr^2 \cdot \left[\frac{\mathrm{d}x_2^2}{1 - x_2^2} + \mathrm{d}x_3^2 \left(1 - x_2^2\right)\right] , \end{aligned} \tag{1.8}$$

what we want to write in the form:

$$\mathrm{d}s^2 = f_4 \cdot \mathrm{d}x_4^2 - f_1 \cdot \mathrm{d}x_1^2 - f_2 \cdot \frac{\mathrm{d}x_2^2}{(1 - x_2^2)} - f_3 \cdot \mathrm{d}x_3^2 \cdot \left(1 - x_2^2\right) . \tag{1.9}$$

In this case f_1, $f_2 = f_3$, f_4 are three functions of x_1, which have to satisfy the following conditions:

1. For $x_1 = \infty$ the following must hold:

$$\begin{aligned} f_1 &= \frac{1}{r^4} = \frac{1}{\sqrt[3]{(3x_1)^4}} \\ f_2 &= f_3 = r^2 = \sqrt[3]{(3x_1)^2} \\ f_4 &= 1 , \end{aligned}$$

2. The determinant equation: $f_1 \cdot f_2 \cdot f_3 \cdot f_4 = 1$,

3. The field equations,

4. The function f is continuous everywhere except $x_1 = 0$.

§4. In order to be able to set up the field equations, one must first form the components of the gravitational field corresponding to the line element Equation 1.9. The easiest way to do this is to form the differential equations of the geodetic line by directly executing the variation and reading the components of the former. The differential equations of the geodetic line for the line element according to Equation 1.9 result from the variation directly in the form:

$$
\begin{aligned}
0 &= f_1 \cdot \frac{\mathrm{d}^2 x_1}{\mathrm{d}s^2} + \frac{1}{2} \cdot \frac{\partial f_4}{\partial x_1} \left(\frac{\mathrm{d}x_4}{\mathrm{d}s}\right)^2 + \frac{1}{2} \cdot \frac{\partial f_1}{\partial x_1} \left(\frac{\mathrm{d}x_1}{\mathrm{d}s}\right)^2 \\
&\quad - \frac{1}{2} \cdot \frac{\partial f_2}{\partial x_1} \left[\frac{1}{(1-x_2^2)} \cdot \left(\frac{\mathrm{d}x_2}{\mathrm{d}s}\right)^2 + \left(1-x_2^2\right) \cdot \left(\frac{\mathrm{d}x_3}{\mathrm{d}s}\right)^2 \right] \\
0 &= \frac{f_2}{(1-x_2^2)} \cdot \frac{\mathrm{d}^2 x_2}{\mathrm{d}s^2} + \frac{\partial f_2}{\partial x_1} \frac{1}{(1-x_2^2)} \frac{\mathrm{d}x_1}{\mathrm{d}s} \frac{\mathrm{d}x_2}{\mathrm{d}s} \\
&\quad + \frac{f_2 \cdot x_2}{(1-x_2^2)^2} \cdot \left(\frac{\mathrm{d}^2 x_2}{\mathrm{d}s^2}\right)^2 + f_2 \cdot x_2 \cdot \left(\frac{\mathrm{d}x_3}{\mathrm{d}s}\right)^2 \\
0 &= f_2 \cdot \left(1-x_2^2\right) \cdot \frac{\mathrm{d}^2 x_3}{\mathrm{d}s^2} + \frac{\partial f_2}{\partial x_1} \left(1-x_2^2\right) \frac{\mathrm{d}x_1}{\mathrm{d}s} \frac{\mathrm{d}x_3}{\mathrm{d}s} \\
&\quad - 2 f_2 \cdot x_2 \cdot \frac{\mathrm{d}x_2}{\mathrm{d}s} \frac{\mathrm{d}x_3}{\mathrm{d}s} \\
0 &= f_4 \cdot \frac{\mathrm{d}^2 x_4}{\mathrm{d}s^2} + \frac{\partial f_4}{\partial x_1} \frac{\mathrm{d}x_1}{\mathrm{d}s} \frac{\mathrm{d}x_4}{\mathrm{d}s} \,.
\end{aligned}
$$

Comparing these equations with Equation 1.2 gives the components of the gravitational field:

$$
\begin{aligned}
F_{11}^1 &= -\frac{1}{2} \cdot \frac{1}{f_1} \cdot \frac{\partial f_1}{\partial x_1} \,, \\
F_{22}^1 &= +\frac{1}{2} \cdot \frac{1}{f_1} \cdot \frac{\partial f_2}{\partial x_1} \cdot \left(1-x_2^2\right) \,, \\
F_{33}^1 &= +\frac{1}{2} \cdot \frac{1}{f_1} \cdot \frac{\partial f_3}{\partial x_1} \cdot \left(1-x_2^2\right) \,, \\
F_{44}^1 &= -\frac{1}{2} \cdot \frac{1}{f_1} \cdot \frac{\partial f_4}{\partial x_1} \,,
\end{aligned}
$$

$$
\begin{aligned}
F_{21}^2 &= -\frac{1}{2} \cdot \frac{1}{f_2} \cdot \frac{\partial f_2}{\partial x_1} \,, \\
F_{22}^2 &= -\frac{x_2}{(1-x_2^2)} \,, \\
F_{23}^2 &= -x_2 \cdot \left(1-x_2^2\right) \,,
\end{aligned}
$$

$$F_{31}^{3} = -\frac{1}{2}\cdot\frac{1}{f_3}\cdot\frac{\partial f_3}{\partial x_1}\,,$$
$$F_{32}^{3} = +\frac{x_2}{(1-x_2^2)}\,,$$
$$F_{41}^{4} = -\frac{1}{2}\cdot\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\,,$$

while the rest of the components of it are all zero.

Because there exist a rotational symmetry about the origin of the coordinate system, it is sufficient to form the field equations only for the equator ($x_2 = 0$), so that since the differentiation is only carried out once, one can set $(1 - x_2^2)$ equal to 1 everywhere in the above formulæ. The calculation of the field equations then yields the form

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1}\right) = \frac{1}{2}\cdot\left(\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_4}\right)^2 + \frac{1}{2}\cdot\left(\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\right)^2\,, \quad \text{(a)}$$

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_2}{\partial x_1}\right) = 2 + \frac{1}{f_1\cdot f_2}\cdot\left(\frac{\partial f_2}{\partial x_1}\right)^2\,, \quad \text{(b)}$$

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_4}{\partial x_1}\right) = \frac{1}{f_1\cdot f_4}\cdot\left(\frac{\partial f_4}{\partial x_1}\right)^2\,. \quad \text{(c)}$$

Besides these three equations, the functions f_1, f_2, f_4 must satisfy the determinant equation

$$f_1\cdot f_2^2\cdot f_4 = 1\,, \quad \text{(d)}$$

or:

$$\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1} + \frac{2}{f_2}\cdot\frac{\partial f_2}{\partial x_1} + \frac{2}{f_4}\cdot\frac{\partial f_4}{\partial x_1} = 0\,.$$

We initially ignore Equation b and determine the three functions f_1, f_2, f_4 from Equation a, Equation c and Equation d. Equation c can be converted into the form

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\right) = \frac{1}{f_1\cdot f_4}\cdot\frac{\partial f_1}{\partial x_1}\frac{\partial f_4}{\partial x_1}\,. \quad \text{(c')}$$

Equation c' may be initially integrated to give

$$\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1} = \alpha\cdot f_1\,, \quad \text{(c")}$$

where α is a constant of integration.

Summation of Equation a and Equation c' gives

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1}+\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\right)=\left(\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_1}\right)^2+\frac{1}{2}\cdot\left(\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1}+\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\right)^2.$$

With taking Equation d into account, it follows that

$$-2\cdot\frac{\partial}{\partial x_1}\left(\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_1}\right)=3\cdot\left(\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_1}\right)^2$$

After integration, we obtain

$$\frac{1}{\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_1}}=\frac{3}{2}\cdot x_1+\frac{\zeta}{2},$$

where ζ is a constant of integration. After a bit rearrangement we obtain

$$\frac{1}{f_2}\cdot\frac{\partial f_2}{\partial x_1}=\frac{2}{(3\cdot x_1+\zeta)}.$$

After further integration we get

$$f_2=\lambda\cdot\sqrt[3]{(3\cdot x_1+\zeta)^2},$$

where λ is a constant of integration.
The condition at infinity requires that $\lambda=1$. Therefore

$$f_2=\sqrt[3]{(3\cdot x_1+\zeta)^2}. \tag{1.10}$$

Next, it follows from Equation c" and Equation d that

$$\frac{\partial f_4}{\partial x_1}=\alpha\cdot f_1\cdot f_4=\frac{\alpha}{f_2^2}=\frac{\alpha}{\sqrt[3]{(3\cdot x_1+\zeta)^2}}.$$

We integrate with respect to the condition at infinity and obtain:

$$f_4=1-\alpha\cdot\frac{1}{\sqrt[3]{(3\cdot x_1+\zeta)}}. \tag{1.11}$$

Finally, it follows from Equation d that

$$f_1=\frac{1}{\sqrt[3]{(3\cdot x_1+\zeta)^4}}\cdot\frac{1}{1-\alpha\cdot\frac{1}{\sqrt[3]{(3\cdot x_1+\zeta)}}}. \tag{1.12}$$

It is easy to check that Equation b is fulfilled within itself with the found formulæ for f_1 and f_2.

With this all requirements are satisfied except for the *continuity condition*. The function f_1 becomes discontinuous when

$$1 = \frac{\alpha}{\sqrt[3]{3 \cdot x_1 + \zeta}}\,,$$
$$3 \cdot x_1 = \alpha^3 - \zeta\,.$$

In order for this discontinuity to coincide with the zero point, the relationship

$$\zeta = \alpha^3\,. \tag{1.13}$$

must hold. In this way the continuity condition links the two integration constants α and ζ together. The complete solution to our problem is now as follows:

$$f_1 = \frac{1}{R^4} \cdot \frac{1}{1 - \frac{\alpha}{R}}\,,$$
$$f_2 = f_3 = R^2\,,$$
$$f_4 = 1 - \frac{\alpha}{R}\,,$$

where the auxiliary quantity R according to

$$R = \sqrt[3]{3x_1 + \zeta} = \sqrt[3]{r^3 + \alpha^3}$$

has been introduced.

If we now substitute these quantities of the above formulæ of the functions f into the formula of the line element according to Equation 1.9, and at the same time turning back to regular spherical coordinates, we arrive at such a *formula for the line element*

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{R}\right)\mathrm{d}t^2 - \frac{\mathrm{d}R^2}{1 - \frac{\alpha}{R}} - R^2 \cdot \left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\phi^2\right)\,, \tag{1.14}$$

which is the exact solution of the Einstein problem and where

$$R = \sqrt[3]{r^3 + \alpha^3}\,.$$

This equation contains the one constant α, which depends on the size of the mass located at the origin.

§5. The uniqueness of this solution follows from the aforementioned calculations. That it would be difficult to recognize the uniqueness from Mr. EINSTEIN's approximative method is seen from the following fact: From the above mentioned calculations it followed, right before the continuity condition was applied, that:

$$f_1 = \frac{1}{\sqrt[4]{(3x_1+\zeta)^3}} \cdot \frac{1}{1-\alpha \cdot \frac{1}{\sqrt[3]{(3x_1+\zeta)}}} = \frac{1}{\sqrt[4]{(r^3+\zeta)^3}} \cdot \frac{1}{1-\alpha \cdot \frac{1}{\sqrt[3]{(r^3+\zeta)}}} \cdot$$

In the case where α and ζ are small values, the TAYLOR series expansion yields, up to terms of the second order, the equation

$$f_1 = \frac{1}{r^4} \cdot \left[1 + \frac{\alpha}{r} - \frac{4}{3} \cdot \frac{\zeta}{r^3}\right] .$$

This expression, together with those correspondingly developed by f_2, f_3, f_4, satisfies all the requirements of the problem within the same accuracy. The continuity requirement does not add anything new within this approximation, since discontinuities only appear by themselves at the origin. So, the two constants α and ζ appear to remain arbitrary, which would make the problem physically undetermined. The sound solution shows that in reality, when proceeding with the approximations, the discontinuity does not occur at the origin, but at the point $r = \sqrt[3]{\alpha^3 - \zeta}$, and that one has to set $\zeta = \alpha^3$ so that the discontinuity tends to the origin. When approaching powers of α and ζ, one would have to have a very good overview of the law of coefficients in order to recognize the necessity of this connection between α and ζ.

§6. Finally, the *movement of a point in the gravitational field*, the geodetic line belonging to the line element Equation 1.14, is to be derived. From the three facts that the line element is homogeneous in the differentials and its coefficients are independent of t and ζ, the variation immediately results in three intermediate integrals. If one restricts oneself to the movement in the equatorial plane ($\vartheta = 90°, \mathrm{d}\vartheta = 0$), these intermediate integrals are written in the form

$$\left(1 - \frac{\alpha}{R}\right) \cdot \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{1}{1-\frac{\alpha}{R}} \cdot \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)^2 - R^2 \cdot \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = \text{ const. } = h\,, \qquad (1.15)$$

$$R^2 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}s} = \text{ const. } = c\,, \qquad (1.16)$$

$$\left(1 - \frac{\alpha}{R}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}s} = \text{ const. } = 1 \quad \text{(Determination of the time interval).} \qquad (1.17)$$

From this it follows that

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\phi}\right)^2 + R^2 \cdot \left(1 - \frac{\alpha}{R}\right) = \frac{R^4}{c^2} \cdot \left[1 - h \cdot \left(1 - \frac{\alpha}{R}\right)\right]$$

or for $1/R = x$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 = \frac{1-h}{c^2} + \frac{h \cdot \alpha}{c^2} \cdot x - x^2 + \alpha \cdot x^3 \,. \tag{1.18}$$

If one introduces the notations $\frac{c^2}{h} = B$, $\frac{(1-h)}{h} = 2 \cdot A$, this is identical to Mr. Einstein's Equation 1.11 a.a.0. and reflects the observed anomaly of Mercury's perihelion.
In general, Mr. Einstein's approximation for the trajectory turns into the sound solution if one only introduces the quantity

$$R = \sqrt[3]{r^3 + \alpha^3} = r \cdot \sqrt[3]{\left(1 + \frac{\alpha^3}{r^3}\right)} \tag{1.19}$$

instead of r. Since $\frac{\alpha}{r}$ is almost equal to twice the square of the planet's velocity (unit in speed of light), the bracket in Equation 1.19 itself for Mercury is only different from 1 by magnitudes of order 1×10^{-12}. So R is practically identical to r and Mr. Einstein's approximation is sufficient for most of all practical needs.
Finally, the sound form of Kepler's third law for circular orbits shall be derived. According to Equation 1.16 and Equation 1.17, if one introduces $x = 1/R$, the following applies to the angular velocity $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$:

$$\omega = c \cdot x^2 \cdot (1 - \alpha \cdot x) \,.$$

For circular orbits, both $\frac{\mathrm{d}x}{\mathrm{d}\phi}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2}$ must be zero. According to Equation 1.18 this results in

$$\begin{aligned} 0 &= \frac{(1-h)}{c^2} + \frac{h \cdot \alpha}{c^2} \cdot x - x^2 + \alpha \cdot x^3 \,, \\ 0 &= \frac{h \cdot \alpha}{c^2} - 2x + 3\alpha \cdot x^2 \,. \end{aligned}$$

Eliminating h from these two equations gives

$$\alpha = 2c^2 x \cdot (1 - \alpha \cdot x)^2 \,.$$

This means that

$$\omega^2 = \frac{\alpha}{2} \cdot x^3 = \frac{\alpha}{2 \cdot R^3} = \frac{\alpha}{2 \cdot (r^3 + \alpha^3)} \,.$$

Up to the surface of the sun, the deviation of this formula from Kepler's third law is completely imperceptible. For an ideal mass point, however, it follows that

the angular velocity does not increase indefinitely as the orbit radius is reduced, as with NEWTON's law, but rather approaches a certain limit, namely

$$\omega_0 = \frac{1}{\alpha \cdot \sqrt{2}} \,.$$

(For a point on the order of the mass of the sun, the cutoff frequency is around 1×10^4 per second.) If similar laws apply to molecular forces, this circumstance may be of some interest.

Part II.

Extended Body in Space-Time – Sphere

2. On the gravitational field of a sphere of incompressible fluid according to Einstein's theory

§1. As a further example of EINSTEIN's theory of gravitation I have calculated the gravitational field of a homogeneous sphere of finite radius, which consists of incompressible fluid. The addition "of incompressible fluid" is necessary, since in the theory of relativity gravitation depends not only on the quantity of matter, but also on its energy, and e.g. a solid body in a given state of tension would result in a gravitational field different from that of a fluid.

The computation is an immediate extension of my communication on the gravitational field of a mass point (see Part I), that I shall quote as "*Mass point*" for short.

§2. EINSTEIN's field equations of gravitation, [2], are stated in general in the form:

$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = G_{\mu\nu}\,. \tag{2.1}$$

The quantities $G_{\mu\nu}$ vanish where no matter is present. In the interior of an incompressible fluid they are determined in the following way: the "mixed energy tensor" of an incompressible fluid at rest is, according to Mr. EINSTEIN, [1], (Note that the $\mathbf{P}$ present there vanishes due to the incompressibility):

$$\left.\begin{aligned} &T^1_1 = T^2_2 = T^3_3 = -p\,, \\ &T^4_4 = \rho_0\,, (\text{the remaining } T^{\nu}_{\mu} = 0)\,. \end{aligned}\right\} \tag{2.2}$$

Here p means the pressure, ρ_0 the constant density of the fluid.

The "covariant energy tensor" will become:

$$T_{\mu\nu} = \sum_{\tau} T^{\tau}_{\mu} g_{\nu\tau}\,. \tag{2.3}$$

Furthermore:

$$T = \sum_{\tau} T^{\tau}_{\tau} = \rho_0 - 3 \cdot p \tag{2.4}$$

and

$$\varkappa = 8\pi \cdot k^2 \,,$$

where k^2 is the GAUSSian gravitational constant. Then, according to Mr. EINSTEIN, see [2], the right-hand sides of the field equations read as:

$$G_{\mu\nu} = -\varkappa \cdot \left(T_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} T \right) . \tag{2.5}$$

For that the fluid is in equilibrium, the following conditions must be satisfied (ibidem Eq. 7a[1]):

$$\sum_{\alpha} \frac{\partial T_{\tau}^{\alpha}}{\partial x_{\alpha}} + \sum_{\mu\nu} \Gamma_{\tau\nu}^{\mu} T_{\mu}^{\nu} = 0 \,. \tag{2.6}$$

§3. Just as in the case of the mass point, for the sphere the general equations must also be specified for the case of rotational symmetry about the origin. Like there, it is convenient to introduce the polar coordinates of determinant 1:

$$\left.\begin{aligned} x_1 &= \frac{r^3}{2} \,, \\ x_2 &= -\cos(\vartheta) \,, \\ x_3 &= \phi \,, \\ x_4 &= t \,. \end{aligned}\right\} \tag{2.7}$$

Then the line element, like there, must have the form:

$$\mathrm{d}s^2 = f_4 \cdot \mathrm{d}x_4^2 - f_1 \cdot \mathrm{d}x_1^2 - f_2 \cdot \frac{\mathrm{d}x_2^2}{(1 - x_2^2)} - f_2 \cdot \mathrm{d}x_3^2 \cdot \left(1 - x_2^2\right) \tag{2.8}$$

hence one has:

$$\begin{aligned} g_{11} &= -f_1 \,, \\ g_{22} &= -\frac{f_2}{(1 - x_2^2)} \,, \\ g_{33} &= -f_2 \cdot \left(1 - x_2^2\right) \,, \\ g_{44} &= f_4 \,, \end{aligned}$$

(the remaining $\mathrm{g}_{\mu\nu} = 0$). Moreover the f are functions only of x_1.

[1] see `https://www.biodiversitylibrary.org/item/92536`

The solutions according to Equation 1.10, Equation 1.11, Equation 1.12 also result for the space outside the sphere in the form:

$$\left.\begin{aligned} f_4 &= 1 - \alpha \cdot \frac{1}{\sqrt[3]{3x_1 + \zeta}}\,, \\ f_2 &= \sqrt[3]{(3x_1 + \zeta)^2}\,, \\ f_1 \cdot f_2^2 \cdot f_4 &= 1\,, \end{aligned}\right\} \tag{2.9}$$

where α and ζ are two initially arbitrary constants that still have to be determined from the mass and radius of our sphere we are considering.

It remains the task to establish the field equations for the interior of the sphere by means of Equation 2.8 for the line element, and to solve them. For the right-hand sides one obtains in sequence:

$$\begin{aligned} T_{11} &= T_1^1 g_{11} = -p \cdot f_1\,, \\ T_{22} &= T_2^2 g_{22} = -\frac{p \cdot f_2}{(1 - x_2^2)}\,, \\ T_{33} &= T_3^3 g_{33} = -p \cdot f_2 \cdot \left(1 - x_2^2\right) \\ T_{44} &= T_4^4 g_{44} = \rho_0 \cdot f_4\,, \\ G_{11} &= \frac{\varkappa \cdot f_1}{2} \cdot (p - \rho_0)\ , \\ G_{22} &= \frac{\varkappa \cdot f_2}{2} \cdot \frac{1}{(1 - x_2^2)} \cdot (p - \rho_0)\ , \\ G_{33} &= \frac{\varkappa \cdot f_2}{2} \cdot \left(1 - x_2^2\right) \cdot (p - \rho_0)\ , \\ G_{44} &= -\frac{\varkappa \cdot f_4}{2} \cdot (\rho_0 + 3p)\ . \end{aligned}$$

Unchanged by the mass point (see page 15), the expressions of the components $\Gamma^{\alpha}_{\mu\nu}$ of the gravitational field can be taken over by the function f and the left-hand sides of the field equations. We get the following system of equations as a whole if we limit ourself to the equator, i.e. $x_2 = 0$:

First the three field equations:

$$\left.\begin{aligned} &-\frac{1}{2} \cdot \frac{\partial}{\partial x_1}\left(\frac{1}{f_1} \cdot \frac{\partial f_1}{\partial x_1}\right) + \frac{1}{4} \cdot \frac{1}{f_1^2} \cdot \left(\frac{\partial f_1}{\partial x_1}\right)^2 + \frac{1}{2} \cdot \frac{1}{f_2^2} \cdot \left(\frac{\partial f_2}{\partial x_1}\right)^2 \\ &+ \frac{1}{4} \cdot \frac{1}{f_4^2} \cdot \left(\frac{\partial f_4}{\partial x_1}\right)^2 = -\frac{\varkappa}{2} \cdot f_1 \cdot (\rho_0 - p)\ , \end{aligned}\right\} \tag{2.10a}$$

$$\left.\begin{aligned} &+\frac{1}{2}\cdot\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_2}{\partial x_1}\right)-1-\frac{1}{2}\cdot\frac{1}{f_1\cdot f_2}\cdot\left(\frac{\partial f_2}{\partial x_1}\right)^2 \\ &=-\frac{\varkappa}{2}\cdot f_2\cdot(\rho_0-p)\ , \end{aligned}\right\} \tag{2.10b}$$

$$\left.\begin{aligned} &-\frac{1}{2}\cdot\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\cdot\frac{\partial f_4}{\partial x_1}\right)+\frac{1}{2}\cdot\frac{1}{f_1\cdot f_4}\cdot\left(\frac{\partial f_4}{\partial x_1}\right)^2 \\ &=-\frac{\varkappa}{2}\cdot f_4\cdot(\rho_0+3p)\ . \end{aligned}\right\} \tag{2.10c}$$

In addition comes the equation for the determinant:

$$f_1\cdot f_2^2\cdot f_4=1\,. \tag{2.10d}$$

The equilibrium conditions according to Equation 2.6 yield the single equation:

$$-\frac{\partial p}{\partial x_1}=-\frac{p}{2}\cdot\left[\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial x_1}+\frac{2}{f_2}\cdot\frac{\partial f_2}{\partial x_1}\right]+\frac{\rho_0}{2}\cdot\frac{1}{f_4}\cdot\frac{\partial f_4}{\partial x_1}\,. \tag{2.10e}$$

From the general considerations of Mr. Einstein it turns out that the present five equations with the four unknown functions f_1, f_2, f_4, p are mutually compatible.

We have to determine a solution of these five equations that is free from singularities in the interior of the sphere. At the surface $p=0$ must hold and there the functions f along with their first derivatives must continuously change into the values valid outside the sphere according to Equation 2.9.

For simplicity the index 1 of x_1 will be henceforth omitted.

§4. By means of the equation for the determinant the equilibrium condition Equation 2.10e becomes:

$$-pdvpx = \frac{\rho_0 + p}{2} \cdot \frac{1}{f_4} \cdot \frac{\partial f_4}{\partial x} .$$

This can be immediately integrated and gives:

$$(\rho_0 + p) \cdot \sqrt{f_4} = \text{ const. } = \gamma . \tag{2.11}$$

Through multiplication by the factors -2, $+2 \cdot \frac{f_1}{f_2}$, $-2 \cdot \frac{f_1}{f_4}$ the field equations according to Equation 2.10a, Equation 2.10b, Equation 2.10c transform into:

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{1}{f_1} \cdot \frac{\partial f_1}{\partial x}\right) + \frac{1}{2 \cdot f_1^2} \cdot \left(\frac{\partial f_1}{\partial x}\right)^2 + \frac{1}{f_2^2} \cdot \left(\frac{\partial f_2}{\partial x}\right)^2 \\ &+ \frac{1}{2 \cdot f_4^2} \cdot \left(\frac{\partial f_4}{\partial x}\right)^2 + \varkappa f_1 \cdot (\rho_0 - p) , \end{aligned}\right\} \tag{2.10a'}$$

$$\left.\frac{\partial}{\partial x}\left(\frac{1}{f_2} \cdot \frac{\partial f_2}{\partial x}\right) = 2 \cdot \frac{f_1}{f_2} + \frac{1}{f_1 \cdot f_2} \cdot \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x} - \varkappa f_1 \cdot (\rho_0 - p) ,\right\} \tag{2.10b'}$$

$$\left.\frac{\partial}{\partial x}\left(\frac{1}{f_4} \cdot \frac{\partial f_4}{\partial x}\right) = \frac{1}{f_1 \cdot f_4} \cdot \frac{\partial f_1}{\partial x}\frac{\partial f_4}{\partial x} + \varkappa f_1 \cdot (\rho_0 + 3p) .\right\} \tag{2.10c'}$$

If one builds the combinations Equation 2.10a' plus 2 times Equation 2.10b' plus Equation 2.10c' and Equation 2.10a' plus Equation 2.10b', by availing of the equation for the determinant one gets:

$$0 = 4 \cdot \frac{f_1}{f_2} - \frac{1}{f_2^2} \cdot \left(\frac{\partial f_2}{\partial x}\right)^2 - \frac{2}{f_2 \cdot f_4} \cdot \frac{\partial f_2}{\partial x}\frac{\partial f_4}{\partial x} + 4\varkappa f_1 p \tag{2.12}$$

and

$$0 = 2 \cdot \frac{\partial}{\partial x}\left(\frac{1}{f_2} \cdot \frac{\partial f_2}{\partial x}\right) + \frac{3}{f_2^2} \cdot \left(\frac{\partial f_2}{\partial x}\right)^2 + 2\varkappa f_1 \cdot (\rho_0 + p) . \tag{2.13}$$

We want to introduce new variables here, which prove to be beneficial because, according to the results, they behave very simply at the mass point outside the sphere, so that they also have to reduce the parts of the current equations that do not contain ρ_0 and p to a simple form.

Let it know be:

$$\left.\begin{aligned} f_2 &= \eta^{\frac{2}{3}}\,, \\ f_4 &= \xi \cdot \eta^{-\frac{1}{3}}\,, \\ f_1 &= \frac{1}{\xi \cdot \eta}\,. \end{aligned}\right\} \tag{2.14}$$

Then, according to Equation 2.9, outside the sphere the following holds:

$$\left.\begin{aligned} \eta &= 3x + \rho\,, \\ \xi &= \eta^{\frac{1}{3}} - \alpha\,, \end{aligned}\right\} \tag{2.15}$$

$$\left.\begin{aligned} \frac{\partial \eta}{\partial x} &= 3\,, \\ \frac{\partial \xi}{\partial x} &= \eta^{-\frac{2}{3}}\,. \end{aligned}\right\} \tag{2.16}$$

If one introduces these new variables and at the same time replaces $\rho_0 + p$ by $\gamma \cdot f_4^{-\frac{1}{2}}$ according to Equation 2.11, then Equation 2.12 and Equation 2.13 transform into:

$$\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} = 3 \cdot \eta^{-\frac{2}{3}} + 3\varkappa\gamma \cdot \xi^{-\frac{1}{2}} \cdot \eta^{\frac{1}{6}} - 3\varkappa\rho_0 \tag{2.17}$$

$$2\xi \cdot \frac{\partial^2 \eta}{\partial x^2} = -3\varkappa\gamma \cdot \xi^{-\frac{1}{2}} \cdot \eta^{\frac{1}{6}}\,. \tag{2.18}$$

Adding these two equations gives:

$$2\xi \cdot \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} = 3 \cdot \eta^{-\frac{2}{3}} - 3\varkappa\rho_0\,.$$

The integrating multiplier of this equation is $\frac{\partial \eta}{\partial x}$. The integration of these provides:

$$\xi \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 = 9 \cdot \eta^{\frac{1}{3}} - 3\varkappa\rho_0 \cdot \eta + 9\lambda\,, \tag{2.19}$$

where λ is a constant of integration.
When raised to the 3/2-th power, this gives:

$$\xi^{\frac{3}{2}} \cdot \left(\frac{\partial \eta}{\partial x}\right)^3 = \sqrt{\left(9\eta^{\frac{1}{3}} - 3\varkappa\rho_0 \cdot \eta + 9\lambda\right)^3}\,.$$

If one divides Equation 2.18 by this equation, ξ is cancelled out, and the following

differential equation η remains:

$$\frac{2 \cdot \frac{\partial^2 \eta}{\partial x^2}}{\left(\frac{\partial \eta}{\partial x}\right)^3} = -\frac{3\varkappa\gamma\eta^{\frac{1}{6}}}{\sqrt{\left(9\eta^{\frac{1}{3}} - 3\varkappa\rho_0\eta + 9\lambda\right)^3}}\,.$$

Here, $\frac{\partial \eta}{\partial x}$ is again the integrating multiplier. The integration gives:

$$\frac{2}{\left(\frac{\partial \eta}{\partial x}\right)} = 3\varkappa\gamma \cdot \int \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(9\eta^{\frac{1}{3}} - 3\varkappa\rho_0\eta + 9\lambda\right)^3}} \tag{2.20}$$

and since:

$$\frac{2}{\left(\frac{\partial \eta}{\partial x}\right)} = \frac{2 \cdot \delta x}{\delta \eta}\,,$$

through a further integration it follows that:

$$x = \frac{\varkappa\gamma}{18} \cdot \int \mathrm{d}\eta \cdot \int \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^3}}\,. \tag{2.21}$$

From here x turns out as a function of η, and, vice versa, that η as a function of x. Furthermore ξ follows from Equation 2.19 and Equation 2.20, and hence according to Equation 2.14 the functions f. Finally, our problem is reduced to quadratures.

§5. The integration constants must now be determined in such a way that the interior of the sphere remains free from singularities and that the surface of the sphere is constantly connected to the external values of the functions f and their derivatives, i.e. functions derived using differential calculus.

Let us put $r = r_a$, $x = x_a$, $\eta = \eta_a$, etc. at the surface of the sphere. The continuity of η and ξ can always be maintained by subsequently, appropriately determining the constants α and ζ in Equation 2.15. In order that also the derivatives remain continuous and that according to Equation 2.16

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)_a = 3$$

and

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)_a = \eta_a^{-\frac{2}{3}}$$

will hold good, it must follow from Equation 2.17 and Equation 2.19 that:

$$\left.\begin{aligned} \gamma &= \rho_0 \cdot \xi_a^{\frac{1}{2}} \cdot \eta_a^{-\frac{1}{6}}, \\ \xi_a &= \eta_a^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3} \cdot \eta_a + \lambda. \end{aligned}\right\} \tag{2.22}$$

From this there follows:

$$\xi_a \cdot \eta^{-\frac{1}{3}} = (f_4)_a = 1 - \frac{\varkappa\rho_0}{3} \cdot \eta_a^{\frac{2}{3}} + \lambda \cdot \eta_a^{-\frac{1}{3}}.$$

So as a consequence

$$\gamma = \rho_0 \cdot \sqrt{(f_4)_a}. \tag{2.23}$$

One may see from the comparison with Equation 2.11 that in this way also the condition $p = 0$ at the surface of the sphere is satisfied. The requirement $\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)_a = 3$ provides the following determination for the integration limits in Equation 2.20, namely:

$$3 \cdot \frac{\mathrm{d}x}{\mathrm{d}\eta} = 1 - \frac{\varkappa\gamma}{6} \cdot \int_\eta^{\eta_a} \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3} \cdot \eta + \lambda\right)^3}} \tag{2.24}$$

and thus Equation 2.21 gives the following determination of the integration limits:

$$\left.\begin{aligned} 3 \cdot (x - x_a) &= \eta - \eta_a \\ &\quad + \frac{\varkappa\gamma}{6} \cdot \int_\eta^{\eta_a} \mathrm{d}\eta \cdot \int_\eta^{\eta_a} \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^3}}. \end{aligned}\right\} \tag{2.25}$$

The surface conditions are in this way completely fulfilled. Still undetermined are the two constants η_a and λ, which are fixed by the conditions of continuity at the origin.

We must first of all require that for $x = 0$ it should also be $\eta = 0$. If this were not the case, f_2 would be a finite quantity in the origin, and a change in the angular velocity $\mathrm{d}\phi = \mathrm{d}x_3$ in the coordinate origin, which in reality does not mean any movement at all, would add a contribution to the line element. Hence the condition for determining η_a follows from equation Equation 2.25:

$$3 \cdot x_a = \eta_a - \frac{\varkappa\gamma}{6} \cdot \int_0^{\eta_a} \mathrm{d}\eta \cdot \int_\eta^{\eta_a} \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^3}} \tag{2.26}$$

λ is finally determined by the requirement that the value of the pressure p in the center of the sphere should remain finite and positive, from which it follows

according to Equation 2.11 that at this point f_4 must remain finite and different from zero. Then, according to Equation 2.14, Equation 2.19 and Equation 2.24:

$$\left.\begin{aligned} f_4 = \xi \cdot \eta^{-\frac{1}{3}} = \left(1 - \frac{\varkappa\rho_0}{3} \cdot \eta^{\frac{2}{3}} + \lambda \cdot \eta^{-\frac{1}{3}}\right) \\ \cdot \left[1 - \frac{\varkappa\gamma}{6} \cdot \int_\eta^{\eta_0} \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^3}}\right]^2, \end{aligned}\right\} \tag{2.27}$$

It is first assumed that $\lambda \gtrless 0$. Then follows for a very small η the equation:

$$f_4 \cdot \frac{\lambda}{\sqrt[3]{\eta}} \cdot \left[K + \frac{\varkappa\gamma}{7} \cdot \frac{\eta^{\frac{7}{6}}}{\gamma^{\frac{3}{2}}}\right]^2,$$

where

$$K = 1 - \frac{\varkappa\gamma}{6} \cdot \int_0^{\eta_0} \frac{\eta^{\frac{1}{6}}\,\mathrm{d}\eta}{\sqrt{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3} \cdot \eta + \lambda\right)^3}} \tag{2.28}$$

is set. In the center ($\eta = 0$) f_4 will then be infinite, unless $K = 0$. But, if $K = 0$, f_4 vanishes for $\eta = 0$. In no case, there results for $\eta = 0$ a finite and non-zero f_4. This shows that the requirement $\lambda \gtrless 0$ leads to physically unusable solutions and it follows that $\lambda = 0$.

§6. With the condition $\lambda = 0$ all the integration constants are now determined. At the same time the integrations to be executed become very easy. If one introduces a new variable χ instead of η according to the definition:

$$\sin(\chi) = \sqrt{\frac{\varkappa\rho_0}{3}} \cdot \eta_a^{\frac{1}{3}} \tag{2.29}$$

and analogously

$$\sin(\chi_a) = \sqrt{\frac{\varkappa\rho_0}{3}} \cdot \eta^{\frac{1}{3}}\,, \tag{2.30'}$$

Equation 2.14, Equation 2.27, Equation 2.11, Equation 2.25, and Equation 2.26 transform via an elementary calculation into the following form:

$$\left.\begin{aligned} f_2 &= \frac{3}{\varkappa\rho_0} \cdot \sin^2(\chi)\,, \\ f_4 &= \left(\frac{3 \cdot \cos(\chi_a) - \cos(\chi)}{2}\right)^2\,, \\ f_1 \cdot f_2^2 \cdot f_4 &= 1\,, \end{aligned}\right\} \tag{2.30}$$

$$\rho_0 + p = \rho_0 \cdot \frac{2 \cdot \cos(\chi_a)}{(3 \cdot \cos(\chi_a) - \cos(\chi))}\,, \tag{2.31}$$

$$3x = r^3 = \frac{1}{\sqrt{\left(\frac{\varkappa\rho_0}{3}\right)^3}} \cdot \left[\frac{9}{4} \cdot \cos(\chi_a) \cdot \left(\chi - \frac{1}{2} \cdot \sin(2\chi)\right) - \frac{1}{2} \cdot \sin^3(\chi)\right]\,. \tag{2.32}$$

The constant χ_a is determined through the density ρ_0 and by the radius r_a of the sphere according to the relation:

$$\sqrt{\left(\frac{\varkappa\rho_0}{3}\right)^3} \cdot r_a^3 = \frac{9}{4} \cdot \cos(\chi_a) \cdot \left(\chi_a - \frac{1}{2} \cdot \sin(2\chi_a)\right) - \frac{1}{2} \cdot \sin^3(\chi_a)\,. \tag{2.33}$$

The constants α and ζ of the solution for the outer region follow from condition Equation 2.15 as:

$$\begin{aligned} \zeta &= \eta_a - 3 \cdot x_a\,, \\ \alpha &= \eta_a^{\frac{1}{3}} - \xi_a \end{aligned}$$

and obtain the values:

$$\zeta = \frac{1}{\sqrt{\left(\frac{\varkappa\rho_0}{3}\right)^3}} \cdot \left[\frac{3}{2} \cdot \sin^3(\chi_a) - \frac{9}{4} \cdot \cos(\chi_a) \cdot \left(\chi_a - \frac{1}{2} \cdot \sin(2\chi_a)\right)\right] \tag{2.34}$$

$$\alpha = \frac{1}{\sqrt{\frac{\varkappa\rho_0}{3}}} \cdot \sin^3(\chi_a)\,. \tag{2.35}$$

If one uses the variables χ, ϑ, ϕ instead of x_1, x_2, $x_3(ix)$, the *line element inside the sphere takes on the simple shape*

$$\left.\begin{aligned} \mathrm{d}s^2 = &\left(\frac{3\cdot\cos(\chi_a) - \cos(\chi)}{2}\right)^2 \mathrm{d}t^2 \\ &- \frac{3}{\varkappa\rho_0} \cdot \left[\mathrm{d}\chi^2 + \sin^2(\chi)\,\mathrm{d}\vartheta^2 + \sin^2(\chi)\cdot\sin^2(\vartheta)\,\mathrm{d}\phi^2\right]\,. \end{aligned}\right\} \tag{2.36}$$

Outside the sphere the shape of the line element remains the same as in case of a mass point:

$$\left.\begin{aligned} \mathrm{d}s^2 = &\left(1 - \frac{\alpha}{R}\right)\mathrm{d}t^2 - \frac{\mathrm{d}R^2}{1-\frac{\alpha}{R}} \\ &- R^2\cdot\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\phi^2\right)\,, \end{aligned}\right\} \tag{2.37}$$

where

$$R^3 = r^3 + \zeta\,.$$

Now ζ is determined according to Equation 2.34, while $\zeta = \alpha^3$ applied for the mass point.

§7. The following remarks are linked to the complete solution to our problem contained in the previous paragraph.

7.1. The *spatial* line element ($\mathrm{d}t = 0$) in the interior of the sphere reads:

$$-\,\mathrm{d}s^2 = \frac{3}{\varkappa\rho_0} \cdot \left[\mathrm{d}\chi^2 + \sin^2(\chi)\,\mathrm{d}\vartheta^2 + \sin^2(\chi) \cdot \sin^2(\vartheta)\,\mathrm{d}\phi^2\right] .$$

This is the well-known line element of the so called non-Euclidean geometry of the spherical space. Therefore *the geometry of spherical space prevails inside the sphere* we are considering. The radius of curvature of the spherical space results in

$$\sqrt{\frac{3}{\varkappa\rho_0}}\,.$$

The sphere we are considering does not form the whole of the spherical space, but only part of it, since χ cannot grow to $\frac{\pi}{2}$, but only to the limit χ_a. For the sun, the radius of curvature of the spherical space that dominates the geometry inside would be around 500 times the sun's radius (see the formulæ Equation 2.40 and Equation 2.43).

It is an interesting result of Einstein's theory that it requires reality within spheres exposed to gravity for the geometry of spherical space, which up to now has only had meaning as a possibility, essentially in the form of an idea.

Within the sphere, "naturally measured" lengths are the following quantities:

$$\left.\begin{aligned} &\sqrt{\frac{3}{\varkappa\rho_0}}\,\mathrm{d}\chi\,, \\ &\sqrt{\frac{3}{\varkappa\rho_0}} \cdot \sin(\chi)\,\mathrm{d}\vartheta\,, \\ &\sqrt{\frac{3}{\varkappa\rho_0}} \cdot \sin(\chi) \cdot \sin(\vartheta)\,\mathrm{d}\phi\,. \end{aligned}\right\} \tag{2.38}$$

The radius measured "inside" from the center of the sphere to its surface becomes:

$$P_i = \sqrt{\frac{3}{\varkappa\rho_0}} \cdot \chi_a\,. \tag{2.39}$$

The circumference of the sphere, along a meridian (or any other largest circle) measured and divided by 2π, should be referred to as the "externally measured" radius. It follows:

$$P_a = \sqrt{\frac{3}{\varkappa\rho_0}} \cdot \sin(\chi_a)\,. \tag{2.40}$$

According to Equation 2.37 of the line element outside the sphere this P_a is clearly identical with the value $R_a = \sqrt[3]{r_a^3 + \zeta}$ that the variable R takes at the surface of the sphere.

With the radius P_a one gets for α from Equation 2.35 the simple relations:

$$\left.\begin{aligned} \frac{\alpha}{P_a} &= \sin^2(\chi_a)\,, \\ \alpha &= \frac{\varkappa\rho_0}{3} \cdot P_a^3\,. \end{aligned}\right\} \tag{2.41}$$

So the volume of our sphere becomes:

$$\begin{aligned} V &= \left(\sqrt{\frac{3}{\varkappa\rho_0}}\right)^3 \cdot \int_0^{\chi_a} \mathrm{d}\chi \sin^2(\chi) \cdot \int_0^{\pi} \mathrm{d}\vartheta \sin(\vartheta) \cdot \int_0^{2\pi} \mathrm{d}\phi \\ &= 2\pi \cdot \left(\sqrt{\frac{3}{\varkappa\rho_0}}\right)^3 \cdot \left(\chi_a - \frac{1}{2} \cdot \sin(2\chi_a)\right)\,. \end{aligned}$$

Hence the mass M of our sphere will therefore be ($\varkappa = 8\pi \cdot k^2$)

$$\left.\begin{aligned} M &= \rho_0 \\ V &= \frac{3}{4k^2} \cdot \sqrt{\frac{3}{\varkappa\rho_0}} \cdot \left(\chi_a - \frac{1}{2} \cdot \sin(2\chi_a)\right)\,. \end{aligned}\right\} \tag{2.42}$$

7.2. The following remarks are deduced from the equations of motion of a point of infinitely small mass outside the sphere under consideration, which have the analogous form to that of the mass point (see Equation 2.16, Equation 2.17 and Equation 2.18) there):

At a great distance the movement of the point occurs according to NEWTON's law, where $\frac{\alpha}{2k^2}$ plays the role of the attractive mass M. The value $\frac{\alpha}{2k^2}$ can therefore be referred to as the "gravitational mass" of our sphere.

Furthermore, if a point at infinity is allowed to fall from rest to the surface of the sphere, the "naturally measured" falling speed is given the value:

$$v_a = \frac{1}{\sqrt{1 - \frac{\alpha}{R}}} \cdot \frac{\mathrm{d}R}{\mathrm{d}s} = \sqrt{\frac{\alpha}{R_a}}\,.$$

So according to Equation 2.41 we have:

$$v_a = \sin(\chi_a)\,. \tag{2.43}$$

For the sun, the rate of fall is around $1/500$ of the speed of light. It is easy to

convince yourself that with the small resulting values of χ_a and $\chi(< \chi_a)$, all of our equations, with the exception of the well-known second-order EINSTEIN effects, merge into those of NEWTON's theory.

7.3. For the ratio of the gravitational mass $\frac{\alpha}{2k^2}$ to the substantial mass M one finds the expression

$$\frac{\alpha}{2k^2 M} = \frac{2}{3} \cdot \frac{\sin^3(\chi_a)}{\left(\chi_a - \frac{1}{2} \cdot \sin(2\chi_a)\right)} . \tag{2.44}$$

As the falling velocity $v_a(= \sin(\chi_a))$ and the mass concentration increase, the ratio of the gravitational mass to the substantial mass decreases. This situation is explained by the fact that, for example, with constant mass and increasing density, the transition to a smaller radius occurs with the release of energy (reduction in temperature through radiation).

7.4. The speed of light in our sphere becomes

$$v = \frac{2}{(3 \cdot \cos(\chi_a) - \cos(\chi))} , \tag{2.45}$$

hence it increases with the value

$$\frac{1}{\cos(\chi_a)}$$

at the surface to the value

$$\frac{2}{(3 \cdot \cos(\chi_a) - 1)}$$

at the center. According to Equation 2.11 and Equation 2.31, the pressure size $\rho_0 + p$ increases proportionally to the speed of light.

At the center of the sphere ($\chi = 0$), the speed of light and pressure take on infinitely large values as soon as $\cos(\chi_a) = \frac{1}{3}$, the falling speed, has become equal to the $\sqrt{\frac{8}{9}}$-th part of the (naturally measured) speed of light. There is therefore a concentration limit beyond which a sphere consisting of incompressible fluid cannot exist. If one wanted to apply our equations to values less than $\frac{1}{3}$, i.e. $\cos(\chi_a) < \frac{1}{3}$, one would get discontinuities even outside the center of the sphere. However, for larger χ_a one can find solutions to the problem that are at least continuous outside the center of the sphere if one goes over to the case $\lambda \gtrless 0$ and fulfills the condition $K = 0$ (Equation 2.28). On the way to these solutions, which are of course physically meaningless since they end in an infinitely large pressure at the center, one can move on to the limiting case of a mass concentrated on a point, and then find the relation $\zeta = \alpha^3$ again, which according to the previous investigation applies to the mass point. It should also be noted here that one can only speak of

a mass point if one uses the variable r, which otherwise strikingly plays no role for the geometry and movement within our gravitational field. For an observer who measures from outside, it follows according to Equation 2.41 that a sphere with a given gravitational mass $\frac{\alpha}{2k^2}$ cannot have a smaller externally measured radius than

$$P_a = \alpha\,.$$

For a sphere made of incompressible fluid, the limit takes a value of $\frac{9}{8\alpha}$. (For the sun α becomes 3 km, for a mass of one gram α becomes 1.5×10^{-28} cm).

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Part III.

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